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FINITE ELEMENT ANALYSIS OF PROBLEMS IN  
ELASTICITY/PLASTICITY WITH SINGULARITIES(U) BRUNEL UNIV  
UXBRIDGE (ENGLAND) INST OF COMPUTATIONAL MATHEMATICS

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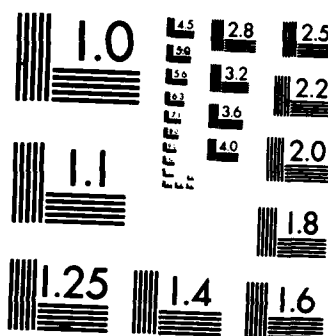
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FINITE ELEMENT ANALYSIS OF PROBLEMS IN  
ELASTICITY/PLASTICITY WITH SINGULARITIES

by

J.R. Whiteman

September 1986

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United States Army

EUROPEAN RESEARCH OFFICE OF THE U.S. ARMY

London England

CONTRACT NUMBER DAJA45-83-C-055 R & D 4238 MA

Institute of Computational Mathematics, Brunel University

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FINITE ELEMENT ANALYSIS OF PROBLEMS IN ELASTICITY/PLASTICITY  
WITH SINGULARITIES

1. AIMS AS STATED IN THE ORIGINAL PROJECT PROPOSAL

The aims of this project, as set out in the original proposal, may be summarized as follows: an investigation would be undertaken into the use of finite element methods for three-dimensional linear problems from potential theory and elasticity containing singularities and for nonlinear problems, particularly those involving the elasto-plastic deformation of three-dimensional solids. Special emphasis would be given to the forms of singularities in three-dimensional linear problems and the use which could be made of these in finite element methods, to the use of complementary energy formulations and variational inequalities in elasto-plastic problems and to the treatment of elasto-plastic problems involving cracks. Throughout consideration would be given to the theoretical basis of the finite element methods developed for these problems, to the development of effective algorithms for their implementation and application to solve practical problems.

These tasks have been adhered to and the results of the research are outlined below. Both linear and nonlinear problems have been studied and these are treated separately in Section 2 (linear) and Section 3 (nonlinear). A total of eleven publications have resulted from this research. These are listed in Section 7 and should be read in conjunction with this Report. ←

2. LINEAR PROBLEMS

As was outlined in the original proposal, whilst many theoretical results exist concerning the form of singularities in two-dimensional problems of potential theory and linear elastic fracture, much less is known concerning the form of singularities for three-dimensional problems of this type. For example in opening mode linear elastic fracture there is still considerable discussion regarding the form of the singularity at the point where a crack front meets a stress free surface. When finite element techniques are used to treat problems involving singularities their numerical efficiency and accuracy depends upon the manner in which the numerical solution is able to model the singular solution. It has long been known that standard finite element techniques do not model singular solutions effectively, so that many special adaptations have been made to improve this situation. Whatever special technique is used, it relies on some knowledge of the form of the singularity. Thus as in the above three-dimensional contexts, when this form is not known it is not possible to produce an effective finite element adaptation with error analysis for the singular problem.

One major feature of the work of this project has been the theoretical analysis of three-dimensional Poisson problems containing re-entrant vertices. In addition to this the use of singular forms in finite element techniques for such problems has been studied. On the theoretical side Beagles and Whiteman have produced two papers [1] and [3] which treat three-dimensional Poisson problems and derive both theoretically and numerically the forms of vertex singularities. For certain problems such as those of Figure 1 the singular forms cannot be derived exactly and numerical approximations have to be obtained. In [1] and [3] numerical schemes are given for obtaining such approximations by solving Laplace-Beltrami problems and eigenvalues, which give the exponent of the radial behaviour, have been calculated together with upper

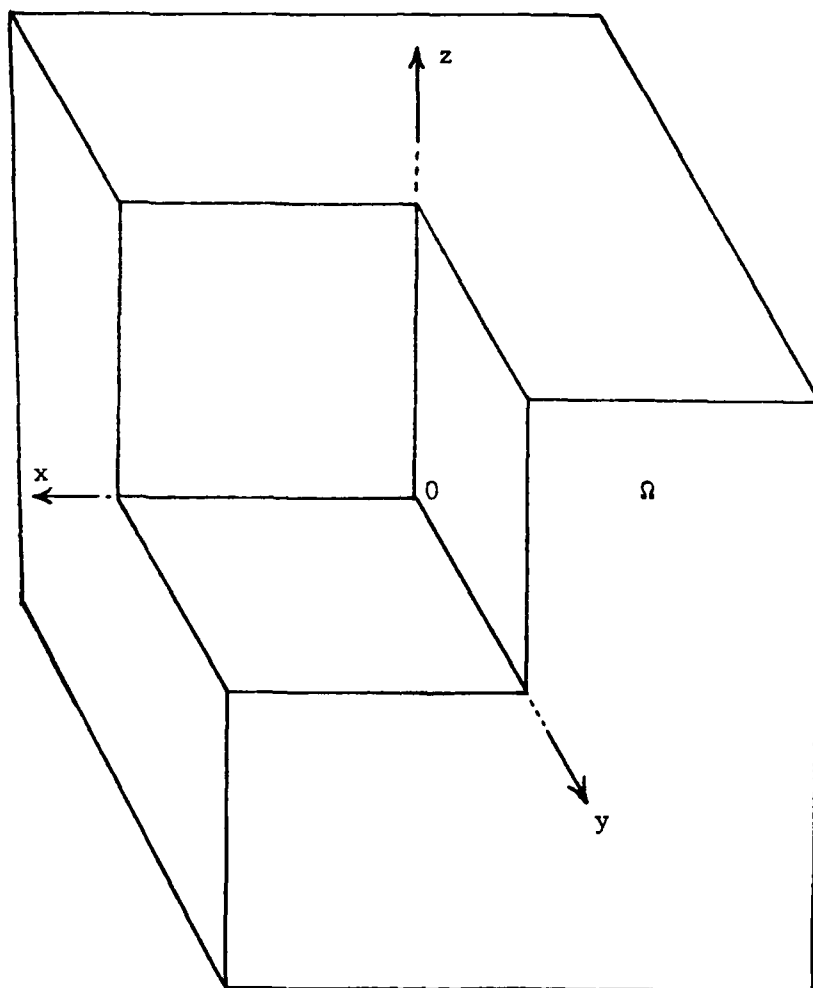


Figure 1

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and lower bounds. When these calculated values have to be used, the singular functions are "non-exact" and so study of the use of non-exact singular functions of this type in finite element techniques has also been made, see [2]. As a result of all this work we now have a much better understanding of the forms of vertex singularities in three-dimensional Poisson problems and their use with finite element techniques. A wide range of vertices has been treated in [3].

In the context of linear elasticity Whiteman and Thompson have studied the stress forms for three-dimensional problems containing cracks and notches and have obtained some preliminary results concerning such forms. This work has again involved the solution of eigenvalue problems and a manuscript [11] has been written. As this subject is one of particular importance and is somewhat an area of dispute, copies of this manuscript are being sent to various colleagues who work in this field, prior to the manuscript being submitted for publication. In summary there are two "camps" in this subject, one of which claims a certain form for the singularity and one of which claims another. Any new paper on the subject is thus bound to be slightly controversial. Once this manuscript is finished it is my intention to incorporate our singular fields into our finite element codes and to undertake analyses similar to those which have been performed for the three-dimensional Poisson problems.

In the context of two- and three-dimensional linear elastic fracture Thompson and Whiteman considered the family of quadratic elements which are modified to produce singular approximating forms by the repositioning of certain nodes; usually on the element edges which intersect at the point of singularity. This is the family of "quarter point elements" and includes transition elements. In [4] a detailed analysis is given of the strain forms produced with these elements at and near crack tips. This indicates the strain forms for various well known elements, particularly radial strains along lines emanating from the point of singularity. The paper [4] sets out to show when the correct singular form is produced and when it is not. It has since been used by Michavila and his co-workers in Madrid [14] as a prototype for the analysis of other singular elements.

#### Superconvergence

An aspect of the finite element treatment of linear problems, which was not envisaged when the proposal for this contract was written and which has emerged during the contract period, is that of superconvergence in finite element techniques.

It has been known for some years that for finite element solutions to Poisson problems and problems of linear elasticity there are specific points (stress points) at which accuracy is better than that found globally. In the last three years several papers have appeared which give theoretical analysis and proofs that gradients can be recovered from finite element solutions to Poisson problems in such a way that at specific points the rate of convergence is higher than that found overall. This is the phenomenon of superconvergence. Unfortunately the main shortcoming of these superconvergence estimates is that they require high global regularity of the solution of the original problem; exactly the property that the solution of a boundary value problem with a singularity does not possess.

In association with Professor M.F. Wheeler of Rice University, I have devoted considerable time to deriving error estimates for the superconvergent recovery of gradients for finite element approximations to Poisson problems in two dimensions. Our major contribution, which appears in [5] concerns the superconvergence on sub-domains of the problems. By deriving results on sub-domains, where the solution has high local regularity, we are able to treat problems involving singularities. We consider the results of [5] to be a very significant advance.

The superconvergence results requiring high global regularity for Poisson problems, which have been published previously in the literature, have now been adapted by Whiteman and Goodsell to the case of linear elasticity and it is intended to adapt these in turn to the case of linear elastic problems involving singularities such as occur in linear elastic fracture. Particularly significant will be the ability to obtain superconvergence results for calculated values of J-integrals. In view of the widespread use of J-integrals as fracture criteria, such an advance would be very important.

### 3. NONLINEAR PROBLEMS

The second major feature of the project has been the treatment of problems of elasto-plasticity. In the proposal a formulation of an elastic-plastic problem in terms of a variational inequality by Oden and Whiteman was set out [12]. This was formulated in terms of complementary energy and was based on the deformation theory of plasticity. One stated possibility was to adapt this to treat problems involving elastic-plastic fracture. In spite of much effort and collaboration between Oden and Whiteman, the complementary energy formulation in terms of stresses proved unsuitable for the fracture context. From the engineering point of view this could perhaps have been expected in view of the shortcomings of deformation theory plasticity. The difficulty stems from the fact that stresses are the primary variables, and for the linear elastic fracture model these are infinite at a crack tip.

In order to treat elasto-plastic fracture problems, we therefore considered incremental plasticity and adapted the MODEL finite element code to treat problems of nonlinear fracture mechanics, see [8]. This code based on the incremental plasticity model works with displacements as primary variables and in the linear elastic context uses a potential energy formulation. Once the code produced reasonable results for displacements in a number of problems, it was extended to calculate integrals of the J-type appropriate to the context of elastic-plastic deformation for problems with cracks. Several papers, [6 - 10] were written on this aspect and calculated values of this path integral were derived. It was found from the numerical results that for the nonlinear case, where significant elastic-plastic deformation takes place, the specific integral appears to be no longer path independent. During a visit to MTL, Watertown, in May 1985 I had lengthy discussions with Dr. D.M. Tracey who confirmed his belief that this effect is correct. Work is now underway to incorporate other integrals of the J-type into the code. These last integrals were proposed in [15] and are path independent in this context.

Thus the current situation is that the MODEL code has been extended to treat nonlinear fracture for problems involving elastic-plastic deformation and has been tested on two-dimensional problems. It has also been applied to a limited number of three-dimensional problems. In this latter case, since integrals of the J-type are two-dimensional phenomena, a family of J values has been

calculated for planes in the solids orthogonal to the crack front. Future work will involve the seeking of a J-type quantity appropriate to nonlinear three-dimensional problems of this type. All this has been done by numerical experiment. At the same time I have been seeking to formulate a rigorous mathematical model of the incremental plasticity problem with hardening, which can be used to provide a theoretical basis with error estimates to the finite element method in this context. Using the analysis of Korneev and Langer [13] I now have one such model. Discussion with Professor J.T. Oden has taken place over this. The next step is to perform the finite element analysis for elastic-plastic problems without singularities and then to extend this to the nonlinear fracture case.

#### 4. CONCLUSIONS

The work of this project may be summarised as follows:

- (a) A considerable number of results have been derived concerning the forms of singularities in three-dimensional Poisson problems.
- (b) Very significant results have been derived for the superconvergent recovery of gradients of finite element solutions to Poisson problems. It is anticipated that these will be important for use in fracture mechanics in the context of the calculation of path integrals; immediately for linear problems but ultimately in nonlinear fracture mechanics.
- (c) Software now exists for the treatment of nonlinear fracture problems and it can be used to calculate J-type integrals in this context. A theoretical model and finite element error estimates are being derived. The next stage will be to derive and compute an integral of the J-type for use on cracks in three-dimensional problems involving elastic-plastic deformation.
- (d) The collaboration with Professor J.T. Oden of the University of Texas proved very useful from the point of view of discussions concerning the treatment of elastic/plastic problems in the context of mathematical formulations for treating plastic behaviour of this type. Extensions to the nonlinear fracture case are now being considered.

#### 5. FUTURE WORK

As is usual at the end of any project, many questions remain unanswered and many new ones have been posed. Clearly exciting opportunities exist for the exploitation of the local superconvergence results. The calculation of J-quantities which are path independent in the nonlinear context and the derivation of an integral for three-dimensional nonlinear fracture are also extremely important. During the course of the project my attention has turned to more complicated materials such as polymers, where phenomena which are time-dependent come in and also where non-isotropic effects are present. I am thus currently studying the fracture of orthotropic materials and also the fracture of materials where viscoelastic and viscoplastic deformation occurs. In this last context I believe it is significant that I have been invited to spend the Fall Semester of 1986 as a Visiting Professor in the Department of Mathematics at Texas A & M University, where I shall collaborate with Dr. J. Walton, and also with Professor R. Shapery, who is the Director of the Mechanics and Materials Center, and who is in addition a world authority on the theory of viscoelastic fracture.



## 6. LECTURES AND VISITS

During the period of this contract I was invited to present talks at the 3rd and 4th US Army Conferences on Applied Mathematics and Computing and I am most grateful to the US Army European Research Office in London for their support which enabled me to attend these conferences. In addition, during the period September 1983 to September 1986, I have visited the Benet Laboratory, Watervliet, and the MTL, Watertown, on several occasions and on each visit have had very useful discussions. A list of the main lectures which I have presented in various parts of the world during the contract period is given in Appendix I.

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APPENDIX 1Main Lectures, September 1983 - September 1986

GAMM Workshop on Numerical Analysis, Kiel	January	1984
MAFELAP 1984, Conference on The Mathematics of Finite Elements and Applications, Brunel University, Uxbridge	May	1984
Summer School of Numerical Analysis, Lancaster	July	1984
Technische Hochschule, Hannover	September	1984
Technische Hochschule, Darmstadt	September	1984
Conference on Mathematical Analysis and Applications, Kuwait	February	1985
Banet Laboratory, Watervliet	May	1985
Department of Computer Science, Rensselaer Polytechnic	May	1985
Third Army Applied Mathematics & Computing Conference, Atlanta	May	1985
Department of Mathematics, University of Texas at Austin	May	1985
Department of Mathematical Sciences, Rice University	May	1985
Department of Mathematics, University of Wales, Aberystwyth	May	1985
Conference on Numerical Analysis, Dundee	June	1985
Conference on Numerical Analysis, Madrid	September	1985
Workshop on Discretisation in Continuum Mechanics, Bad Honnef	September	1985
Material Technology Laboratory, Watertown	January	1986
Department of Mathematics, Texas A & M University	January	1986
Department of Mathematical Sciences, Rice University	January	1986
Department of Mathematics, Herriot Watt University	March	1986
Edinburgh Mathematical Society	March	1986
Department of Mathematics, University of Kuwait	April	1986
Department of Mathematics, University College of Bahrain	April	1986
School of Mathematics, University of Bath	May	1986
Fourth Army Applied Mathematics & Computing Conference, Atlanta	May	1986
Department of Mathematics, University of Strathclyde	June	1986
School of Mining Engineering, University Polytechnic of Madrid	July	1986

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